SOLUTIONS OF PROBLEMS IN NUMBER FIVE.

SOLUTIONS of problems in No. 5 have been received as follows:

From Prof. W. P. Casey, 359, 360, 361; George Eastwood, 361, 363; H. Heaton, 365, 367, A. Hall, Jr., 367, W. E. Heal, 360, 362, 367; Prof. P. H. Philbrick, 362; Prof. E. B. Seitz, 360, 363, 367; Thomas Spencer, 367; Prof. J. Scheffer, 360, 364, 367; R. S. Woodward, 360, 362, 364, 367.

359. "Find the greatest and least number of balls of equal diameter (radius r) that can be put in a given box, a feet long, b feet wide and c feet high."

ANSWER BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Balls whose diameters are equal to the greatest common divisor of a, b and c will give the least number, and those whose diameters are equal to the least c. d. of a, b and c will give the greatest number. The various forms of this question lead to a long discussion.

360. "Prove, of all spherical triangles of equal area, that of the least perimeter is equilateral".

SOLUTION BY PROF. J. SCHEFFER, HARRISBURGH, PA.

Since the area is to be constant, the spherical excess will also be constant, and consequently the three angles; hence, putting

$$A + B + C = 2S, \tag{1}$$

S is constant. Denoting $a+b+c$ by $2s$, we have the formula

$$\tan^2\tfrac{1}{2}s = \tan\tfrac{1}{2}(S-\tfrac{1}{2}\pi)\tan\tfrac{1}{2}(S-C+\tfrac{1}{2}\pi)\tan\tfrac{1}{2}(S-B+\tfrac{1}{2}\pi)\tan\tfrac{1}{2}(S-A+\tfrac{1}{2}\pi)$$

which may, without difficulty, be derived from L'Huilier's formula.

Consequently

$$M = \tan\tfrac{1}{2}(S-C+\tfrac{1}{2}\pi)\tan\tfrac{1}{2}(S-B+\tfrac{1}{2}\pi)\tan\tfrac{1}{2}(S-A+\tfrac{1}{2}\pi) \tag{2}$$

must be a minimum.

Considering A and B the independent variables, we obtain

$$\frac{dM}{dA} = -\frac{1}{2}\frac{\tan\frac{1}{2}(S-C+\frac{1}{2}\pi)}{\cos^2\frac{1}{2}(S-A+\frac{1}{2}\pi)} - \frac{1}{2}\frac{\tan\frac{1}{2}(S-A+\frac{1}{2}\pi)}{\cos^2\frac{1}{2}(S-C+\frac{1}{2}\pi)}\frac{dC}{dA},$$

$$\frac{dM}{dB} = -\frac{1}{2}\frac{\tan\frac{1}{2}(S-C+\frac{1}{2}\pi)}{\cos^2\frac{1}{2}(S-B+\frac{1}{2}\pi)} - \frac{1}{2}\frac{\tan\frac{1}{2}(S-B+\frac{1}{2}\pi)}{\cos^2\frac{1}{2}(S-C+\frac{1}{2}\pi)}\frac{dC}{dB}.$$

From (1) $\qquad \dfrac{dC}{dA} = -1, \quad \dfrac{dC}{dB} = -1.$

Substituting, putting the diff. coef. $= 0$, and clearing of fractions, we obtain the two equations:

$$\sin(S-C+\tfrac{1}{2}\pi) = \sin(S-A+\tfrac{1}{2}\pi),$$
$$\sin(S-C+\tfrac{1}{2}\pi) = \sin(S-B+\tfrac{1}{2}\pi).$$

Wherefore $A = B = C$, and the triangle is equiangular and consequently equilateral.

SOLUTION BY W. E. HEAL, MARION, IND.

The area of a spherical triangle T, whose sides are a, b, c, is

$$\tan\tfrac{1}{4}A(T) = \sqrt{[\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)]}.$$

If the perimeter $P(T)$, $= 2s$, is constant this area will be the greatest when the three factors, $\tan\frac{1}{2}(s-a)$, $\tan\frac{1}{2}(s-b)$, $\tan\frac{1}{2}(s-c)$, are equal; and if we suppose that each side of the triangle is less than π it follows that

$$a = b = c.$$

Now suppose t_1 is an equilateral triangle having the same area as T, that is $A(T) = A(t_1)$. Also let t_2 be an equilateral triangle having the same perimeter as T, that is $P(T) = P(t_2)$. Then by what precedes we must have

$$A(T) < A(t_2),$$
$$\therefore A(t_1) < A(t_2),$$
$$\therefore P(t_1) < P(t_2),$$
$$\therefore P(t_1) < P(T). \quad \text{Q. E. D.}$$

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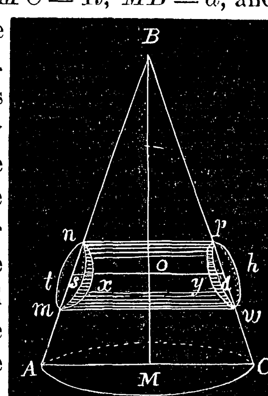
361. "A right cone, radius of base R and altitude a, is pierced by a cylinder whose radius is r, the axis of the cylinder intersecting the axis of the cone at right angles and at a point whose distance from the vertex of the cone is b. Required the solidity common to the cone and cylinder."

SOLUTION BY PROF. CASEY.

Let ABC be the cone, M the center of its base, $MC = R$, $MB = a$, and let $npwm$ represent the solid common to the cone and cylinder; nxm and pyw are equal semi-ellipses.



As $BO = b$, and the radius of the cylinder is given $= r$, $\therefore np$ and mw are known lines, and by well known properties in descriptive geometry the ellipses which form the ends are given. Then if we pass a plane through the element AB perpendicular to the plane ABC and conceive the lateral surface of the cylinder to intersect in the ellipse $nsmt$, and do the same at the other end, forming the ellipse $phwd$, we will have a solid whose ends are plane surfaces bounded by ellipses; and as all the dimen-

sions of this solid are known, its volume is easily found; and as all the dimensions of the ungula *nxmn* are known, its volume may be found; and as there are four of such ungula, subtracting their sum from the volume of *nsmtphwd* we get the required volume.

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362. "A section of an embankment is a feet long; the top width of both ends is b feet; the width of the ends at bottom is c and d feet, respectively, and the corresponding depths of the ends are e and g feet. Develop a Rule, and give the contents."

SOLUTION BY PROF. P. H. PHILBRICK, STATE UNIV., IOWA CITY.

The bottom width at a distance x from first end $= c + x(d-c)\div a$, and the depth at the same point $= e+x(g-e)\div a$. Therefore

$$d\,\text{Vol.} = \tfrac{1}{2}\left[b+c+\frac{x}{a}\left(d-c\right)\right]\left[e+\frac{x}{a}\left(g-e\right)\right]dx$$

$$= \frac{1}{2a^2}\left\{\begin{array}{llll} a^2be + a^2ce & + aedx & + cex^2 \\ & + abgx & + dgx^2 \\ & + acgx & - cgx^2 \\ & - abex & - dex^2 \\ & - 2acex & \end{array}\right\}dx;$$

$$\therefore \text{Vol.} = \frac{1}{2a^2}\left[\begin{array}{l}(a^2be+a^2ce)a+(aed+abg+acg-abe-2ace)\tfrac{1}{2}a^2 \\ +(dg+ce-cg-de)\tfrac{1}{3}a^3\end{array}\right]$$

$$= \tfrac{1}{12}a(3be+3bg+2ce+2dg+ed+cg)$$

$$= \tfrac{1}{12}a[(b+c)e+(b+d)g+(2b+c+d)\,(e+g)].$$

But $(b+c)e =$ twice the area of the first end $= 2A_1$ say, and

$(b+d)g =$ " " second " $= 2A_2$ " .

Again, the width at the middle section $=[\frac{1}{2}(c+d)+b]\div 2=\frac{1}{4}(2b+c+d)$; and the depth $= \frac{1}{2}(e+g)$ and therefore $(2b+c+d)(e+g) =$ eight times the area of the middle section $= 8M$ say.

$$\therefore\ V = \tfrac{1}{12}a(2A_1+8M+2A_2) = \tfrac{1}{6}a(A_1+4M+A_2),$$

which is known as the Prismoidal Formula.

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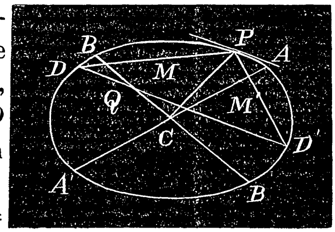
363. "The tangent at one end of a chord of an ellipse is parallel to the line joining the other end with a fixed point within the ellipse. Show that the area of the locus of the middle point of the chord is one half the area of the ellipse."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.



Let C be the center of the ellipse, Q the fixed point, P any point of the ellipse, DD' the chord through Q parallel to the tangent at P, and M, M' the middle points of the chords PD and PD'. Draw the diameter BB' through Q, and the diameter AA' conjugate to BB'.

Let $CA = a$, $CB = b$, $CQ = c$, $\angle ACB = \psi$, and let $(a\cos\varphi, b\sin\varphi)$ be the co-ordinates of P, referred to AA' and BB', (x, y) those of D or D', and (x', y') those of M or M'.

Then the equation of the ellipse is

$$a^2y^2 + b^2x^2 = a^2b^2, \tag{1}$$

and the eq. of DD' is

$$ay\sin\varphi + bx\cos\varphi = ac\sin\varphi. \tag{2}$$

From (1) and (2) we readily find

$$y = c\sin^2\varphi \pm \cos\varphi\sqrt{(b^2 - c^2\sin^2\varphi)},$$
$$bx = ac\sin\varphi\cos \mp a\sin\varphi\sqrt{(b^2 - b^2\sin^2\varphi},$$

the upper sign being taken for D, and the lower for D'.

Hence we have

$$2bx' = b(a\cos\varphi + x) = ab\cos\varphi + ac\sin\varphi\cos\varphi \mp a\sin\varphi\sqrt{(b^2 - c^2\sin^2\varphi)},$$
$$2y' = b\sin\varphi + y = b\sin\varphi + c\sin^2\varphi \pm \cos\varphi\sqrt{(b^2 - c^2\sin^2\varphi)}, \text{ and}$$
$$2dy' = b\cos\varphi\, d\varphi + 2c\sin\varphi\cos\varphi\, d\varphi \mp \sin\varphi\sqrt{(b^2 - c^2\sin^2\varphi)}d\varphi$$
$$\mp c^2\sin\varphi\cos d\varphi \div \sqrt{(b^2 - c^2\sin^2\varphi)}.$$

Therefore the area of the locus of M or the locus of M' is

$$\int x'\sin\psi\, dy' = \int_0^{2\pi}\left\{\tfrac{1}{4}ab\cos^2\varphi + \frac{3ac^2}{4b}\sin^2\varphi\cos^2\varphi + \frac{a}{4b}\sin^2\varphi(b^2 - c^2\sin^2\varphi)\right\}$$
$$\times \sin\psi\, d\varphi,$$

omitting all terms containing odd powers of $\sin\varphi$ or $\cos\varphi$,

$$= \int_0^{2\pi}\left(\tfrac{1}{4}ab + \frac{ac^2}{8b}\cos 2\varphi + \frac{ac^2}{8b}\cos 4\varphi\right)\sin\psi\, d\varphi$$
$$= \tfrac{1}{2}\pi ab\sin\psi = \text{half the area of the ellipse.}$$

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364. "Discuss the curve whose equation is $x = \log[y + \sqrt{(y^2 - 1)}]$, and find its area and length."

SOLUTION BY R. S. WOODWARD, DETROIT, MICH.

Since $x = \log[y + \sqrt{(y^2 - 1)}]$, $y + \sqrt{(y^2 - 1)} = e^{+x}$, and

$$y^2 - 1 = e^{+2x} - 2e^{+x}y + y^2, \text{ whence}$$

$$y = \tfrac{1}{2}(e^{+x} + e^{-x}).$$

This is the equation of a catenary whose directrix is the axis of x. The area lying between the curve and the axis of x for the limits of $+x$ and $-x$ is

$$2\int_0^x ydx = \int_0^x e^{+x}\,dx + e^{-x}\,dx = e^{+x} - e^{-x}.$$

Therefore the area within the curve between the limits $+x$ and $-x$ is

$$x(e^{+x} + e^{-x}) - (e^{+x} - e^{-x})$$
$$= e^{+x}(x-1) + e^{-x}(x+1).$$

The length of the curve between $x = 0$ and $x = \pm x$ is

$$\int_{x=0}^{x=\pm x}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx = \tfrac{1}{2}(e^{\pm x} - e^{\mp x}).$$

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365. "Show that

$$\int_0^{\frac{1}{2}\pi}\frac{\sqrt{(1-c)}\,.\,d\theta}{1-c\cos^n\theta} = \frac{\pi}{\sqrt{(2n)}}$$

when c is indefinitely nearly equal to unity, n being a positive quantity."

SOLUTION BY H. HEATON, LEWIS, IOWA.

$$\int_0^{\frac{1}{2}\pi}\frac{\sqrt{(1-c)}.d\theta}{1-c\cos\theta}$$

$$=\frac{1}{n}\int_0^{\frac{1}{2}\pi}\left(\frac{\sqrt{(1-c)}.d\theta}{1-c^{\frac{1}{n}}\cos\theta}+\frac{\sqrt{(1-c)}[n-1+(n-2)c^{\frac{1}{n}}\cos\theta\ldots+c^{\frac{n-2}{n}}\cos^{(n-2)}\theta]d\theta}{1+c^{\frac{1}{n}}\cos\theta\ldots c^{\frac{n-1}{n}}\cos^{(n-1)}\theta}\right)$$

But the second member on the right-hand of this equa'n $=0$ when $c=1$; because it does not take the form $\frac{0}{0}\,d\theta$ for any value of θ. Hence

$$\int_0^{\frac{1}{2}\pi}\frac{\sqrt{(1-c)}.d\theta}{1-c\cos^n\theta}=\frac{1}{n}\int_0^{\frac{1}{2}\pi}\frac{\sqrt{(1-c)}d\theta}{1-c^{1\div n}\cos\theta}$$

$$=\frac{2\sqrt{(1-c)}}{n\sqrt{(1-c^{1\div n})}}\tan^{-1}\left(\frac{\sqrt{(1-c^{1\div n})}\tan\frac{1}{2}\theta}{\sqrt{(1-c)}}\right)$$

$$=\frac{2\sqrt{(1+c^{1\div n}+c^{2\div n}\ldots+c^{(n-1)\div n})}}{n\sqrt{(1+c^{1\div n})}}\tan^{-1}\left(\frac{\sqrt{(1+c^{1\div n})}\text{tn}\,\frac{1}{2}\theta}{\sqrt{(1-c)}}\right)$$

$$=\frac{\pi}{\sqrt{(2n)}}\text{ when } c=1.$$

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367. "Prove the equation

$$\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) = -\eta^2+\tfrac{1}{2}\eta^4-\tfrac{1}{3}\eta^6+\text{ etc.}$$

$$-2\eta\cos x-\tfrac{1}{2}2\eta^2\cos 2x-\tfrac{1}{3}2\eta^3\cos 3x-\text{etc.}$$

$$=\sum_{i=1}^{i=\infty}(-1)^i\frac{\eta^{2i}}{i}-\sum_{i=1}^{i=\infty}\frac{2\eta^i}{i}\cos ix.$$

SOLUTION BY THOMAS SPENCER, SOUTH MERIDEN, CONN.

From Trigonometry we have the expansion

$$\frac{\eta \sin x}{1-2\eta\cos x+\eta^2} = \eta\sin x + \eta^2\sin 2x + \eta^3\sin 3x + \&c.$$

Multiply both sides of this equation by $2dx$, and integrate, we have

$$\log(1 - 2\eta\cos x + \eta^2) = -2\eta\cos x - \tfrac{1}{2}\,2\eta^2\cos 2x - \tfrac{1}{3}\,2\eta^3\cos 3x - \&c.$$

Also we know that

$$\log(1+\eta^2) = \eta^2 - \tfrac{1}{2}\eta^4 + \tfrac{1}{3}\eta^6 - \&c.$$

Therefore we have

$$\begin{aligned}\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) &= \log(1-2\eta\cos x+\eta^2) - \log(1+\eta^2)\\ &= -\eta^2 + \tfrac{1}{2}\eta^4 - \tfrac{1}{3}\eta^6 + \&c.\\ &\quad - 2\eta\cos x - \tfrac{1}{2}\,2\eta^2\cos 2x - \tfrac{1}{3}\,2\eta^3\cos 3x - \&c.\\ &= \sum_{i=1}^{i=\infty}(-1)^i\frac{\eta^{2i}}{i} - \sum_{i=1}^{i=\infty}\frac{2\eta^i}{i}\cos ix.\end{aligned}$$

SOLUTION BY H. HEATON.

Because $2\cos x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$; therefore

$$\begin{aligned}\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) &= \log\left[1+\eta^2-\eta(e^{x\sqrt{-1}}+e^{-x\sqrt{-1}})\right] - \log(1+\eta^2)\\ &= -\log(1+\eta^2) + \log(1-\eta e^{x\sqrt{-1}}) + \log(1-\eta e^{-x\sqrt{-1}})\\ &= -\eta^2 + \tfrac{1}{2}\eta^4 - \tfrac{1}{3}\eta^6 + \&c.\\ &\quad -2\eta\cos x - \tfrac{1}{2}\,2\eta^2\cos 2x - \tfrac{1}{3}\,2\eta^3\cos 3x - \&c.\end{aligned}$$

PROBLEMS.

368. *By Prof. J. Scheffer.*—In a quadrilateral $ABCD$, the diagonal AC makes with the sides the four angles $CAB = \alpha$, $ACB = \beta$, $ACD = \gamma$, $CAD = \delta$. Find the angles which the other diagonal BD makes with the sides.

369. *By R. J. Adcock.*—Show that the radius of curvature of an ellipse equals the cube of the radius vector divided by the rectangle of the semi axes; the radius vector being through the centre at right angles to the radius of curvature.

370. *By Prof. Edmonds.*—Divide a right angle into three parts α, β, γ, such that $(\cos\alpha) \div m = (\cos\beta) \div n = (\cos\gamma) \div p$.